

A first order phase transition in the threshold $\theta \geq 2$ contact process on random r -regular graphs and r -trees

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Abstract

We consider the discrete time threshold- θ contact process on a random r -regular graph. We show that if $\theta \geq 2$, $r \geq \theta + 2$, ϵ_1 is small and $p \geq p_1(\epsilon_1)$, then starting from all vertices occupied the fraction of occupied vertices is $\geq 1 - 2\epsilon_1$ up to time $\exp(\gamma_1(r)n)$ with high probability. We also show that for $p_2 < 1$ there is an $\epsilon_2(p_2) > 0$ so that if $p \leq p_2$ and the initial density is $\leq \epsilon_2(p_2)$, then the process dies out in time $O(\log n)$. These results imply that the process on the r -tree has a first-order phase transition.

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1. Introduction

The linear contact process was introduced by Harris in 1974 and has been studied extensively since then, see part I of [21]. In that model, the state of the system at time $t \in [0, \infty)$ is $\xi_t : \mathbb{Z}^d \rightarrow \{0, 1\}$, where 1 and 0 correspond to ‘occupied’ and ‘vacant’ respectively. Occupied sites become vacant at rate 1, while a vacant site becomes occupied at rate λk if it has k occupied neighbors.

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In this paper, we will be concerned with particle systems that are versions of the contact process with sexual reproduction. Each site is either occupied (state 1) or vacant (state 0). In our first two models, occupied sites become vacant at rate 1. Perhaps the most natural generalization of the linear contact process is the quadratic contact process in which a vacant site with k occupied neighbors becomes occupied at rate $\lambda \binom{k}{2}$. However, we will primarily be concerned with the threshold- θ contact process in which a vacant site becomes occupied at rate λ if it has $k \geq \theta$ occupied neighbors. The threshold-1 contact process has been studied and found to have the same qualitative behavior as the linear contact process, so we expect that the threshold-2 and quadratic contact processes will behave similarly as well.

Being attractive processes, each of our models with sexual reproduction on translation invariant infinite graphs has a translation invariant upper invariant measure, ξ_∞^1 , that is the limit as $t \rightarrow \infty$ for the system starting from all 1's. See [22,21] for more details about this and the results we cite in the questions below. There are three basic questions for our models.

Q1. Let ξ_t^p be the system starting from product measure with density p , i.e., $\xi_0^p(x)$ are independent and equals 1 with probability p . Does ξ_t^p die out for small p ? That is, do we have $P(\xi_t^p(x) = 1) \rightarrow 0$ as $t \rightarrow \infty$ if $p \leq p_0(\lambda)$?

Q2. Let $\rho(\lambda) = P(\xi_\infty^1(x) = 1)$ and let $\lambda_c = \inf\{\lambda : \rho(\lambda) > 0\}$. Is $\rho(\lambda)$ discontinuous at λ_c ? If so, then soft results imply that $P(\xi_\infty^1(x) = 1) > 0$ when $\lambda = \lambda_c$.

Q3. Let $\xi_\infty^{0,\beta}$ be the limit as $t \rightarrow \infty$ for the system starting from all 0's when sites become occupied spontaneously at rate β along with the original dynamics. Is $\lim_{\beta \rightarrow 0} P(\xi_\infty^{0,\beta}(x) = 1) = 0$? If so, we say that 0 is stable under perturbation, and it follows that there are two nontrivial stationary distributions when $\beta > 0$ is small.

One of the first processes with sexual reproduction that was studied is Toom's NEC (north-east-center) rule on \mathbb{Z}^2 . In its original formulation (see [30,31]) the states of the sites are 1 and -1 . Let e_1, e_2 be the two unit vectors. If the majority of the spins in $\{x, x + e_1, x + e_2\}$ is 1 at time t , then the state of x at time $t + 1$ is 1 with probability $1 - p$ and -1 with probability p . If the majority of the spins is -1 at time t , then the state of x is -1 with probability $1 - q$ and 1 with probability q . If $p + q$ is small, then the system has two stationary distributions; see e.g. [2].

More relevant for us, is the reformulation of Toom's rule as a growth model, where the state of x changes

$$\begin{aligned} 1 &\rightarrow 0 \quad \text{at rate } 1, \\ 0 &\rightarrow 1 \quad \text{at rate } \lambda \text{ if } x + e_1 \text{ and } x + e_2 \text{ are both in state } 1. \end{aligned} \quad (1)$$

For the model in (1),

(a) if we let ξ_t^A denote the set of all 1's at time t starting from $\xi_0^A = A$, and

$$\lambda_f = \inf\{\lambda : P(\xi_t^A \neq \emptyset \text{ for all } t) > 0, \text{ for some finite set } A\}$$

be the critical value for survival from a finite set, then $\lambda_f = \infty$, because if all the 1's in the initial configuration are inside a rectangle, then there will never be any birth of 1's outside that rectangle.

(b) Durrett and Gray [10] used the contour method to prove (see announcement of results in [8]) $\lambda_c \leq 110$.

(c) if p^* is such that $1 - p^*$ equals the critical value for oriented bond percolation on \mathbb{Z}^2 , then for any $p < p^*$ the process starting from product measure with density p dies out.

(d) if $\lambda > \lambda_c$ β is such that $6\beta^{1/4}\lambda^{3/4} < 1$, and if sites become occupied spontaneously at rate β along with the original dynamics, then there are two stationary distributions.

Chen [5,6] has generalized Toom's growth model. He begins by defining the following pairs for each site x .

$$\begin{array}{cccc} \text{pair 1} & \text{pair 2} & \text{pair 3} & \text{pair 4} \\ x - e_1, x - e_2 & x + e_1, x - e_2 & x + e_1, x + e_2 & x - e_1, x + e_2. \end{array}$$

The models are numbered by the pairs that can give birth: Type I (pair 1 = SWC); Type IV (any pair); Type III (pairs 1, 2, and 3); Type 2A (pairs 1 and 2); and Type 2B (pairs 1 and 3). Chen [5] proves for model IV that if $0 < p < p(\lambda)$, then

$$P(0 \in \xi_t^p) \leq t^{-c \log_{2\lambda}(1/p)}.$$

He also shows for the same model that

$$\lim_{\beta \rightarrow 0} P(0 \in \xi_\infty^{0,\beta}) > 0$$

for large λ , so 0 is unstable under perturbation. In contrast, Chen [6] shows that 0 is stable under perturbation in model III.

Durrett and Neuhauser [11] have considered the behavior of the quadratic contact process, with stirring (exchange of values at adjacent sites). In their model, deaths occur at rate 1, and births occur at rate β times the fraction of adjacent pairs that are occupied. The mean field ODE (which assumes adjacent sites are independent) for the density u of 1's in this case:

$$\frac{du}{dt} = -u + \beta(1-u)u^2$$

has $\beta_c = 4$ and $\beta_f = \infty$, where β_c and β_f are analogues of λ_c and λ_f . They have shown that in the limit of fast stirring both critical values converge to 4.5. This threshold arises because depending on whether $\beta > 4.5$ or $\beta < 4.5$, the associated PDE

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} - u + \beta u^2(1-u)$$

has traveling wave solution $u(t, x) = w(x - ct)$ with positive or negative speed c . Based on simulations they have conjectured that the phase transition is continuous.

Evans, Guo, and Liu (see [16–18,24] and [25]) (in various permutations in five papers published in 2007–2009) have considered the quadratic contact process in which particles hop at rate h (i.e., move according to the rules of the simple exclusion process, which for unlabeled particles is the same as stirring). Birth rates are $(1/4)$ times the number of adjacent pairs of occupied sites, deaths occur at rate p . Having $h > 0$ means that the critical value for survival from finite sets $p_f(h) > 0$. When $h < h_0$ is small, $p_f(h) < p_c(h)$, the critical value for the existence of a stationary distribution, and the model has a discontinuous phase transition and 0 is stable under perturbation. When $h \geq h_0$, $p_f(h) = p_c(h)$ and the phase transition is continuous.

The last three authors call their system Schlögl's second model in honor of his (1972) [29] paper which introduced a model with a nonnegative integer number of particles per sites defined by the chemical reactions



i.e., at a site with k particles births occur at rate $c_0 + c_2 \binom{k}{2}$ and deaths occur at rate $c_1 k + c_3 \binom{k}{3}$, and particles jump to a randomly chosen neighbor at rate ν each. The system in which

$$X \rightleftharpoons 2X \quad X \rightleftharpoons 0$$

is Schlögl's first model. It is the analogue of the linear contact process, or if you are a physicist, they are in the same universality class. Grassberger [14] has simulated a version of the second model in which the reaction $3X \rightarrow 2X$ is replaced by the restriction of at most two particles per site, and in which doubly occupied sites give birth onto adjacent sites. He has found that this model has a second order (continuous) phase transition. See also [15], which has been cited more than 300 times, or [28] for a more recent treatment.

The threshold- θ contact process with $\theta \geq 1$ has been studied on \mathbb{Z}^d . Liggett [23] has used it and a comparison to show coexistence in a threshold voter model. See also Chapter II.2 in [21]. Handjani [19] has studied the phase diagram of the model, while [27] have studied asymptotics for its critical values. However, outside the physics literature (see [7]) there are no results about the nature of the phase transition on \mathbb{Z}^d . As we explain later, Fontes and Schonmann [12] have considered the process on a tree.

In this paper, we will consider the discrete time *threshold- θ contact process* on a random r -regular graph, and on trees in which all vertices have degree r . In these processes, sites having at least θ many occupied neighbors at time t become occupied at time $t + 1$ with probability p . Our personal motivation, derived from participating in the 2010–2011 SAMSI program on Complex Networks, is that a random r -regular graph is a toy model for a social network. This model, like the original small world graph of [32], is unrealistic because all vertices have the same number of neighbors. We do not expect the qualitative behavior to change on an Erdős–Rényi graph, but this graph looks locally like a Galton–Watson tree which makes the proofs considerably more complicated.

To see that properties of the model are sensitive to the degree distribution, recall that [4] have shown that if one studies the linear contact process on a random graph with a power law degree distribution, where the degree of a typical vertex is k with probability $p_k \sim Ck^{-\alpha}$, then the critical value is 0 for any $\alpha < \infty$. It is an interesting question to determine whether or not the linear contact process has positive critical value when the degree distribution has an exponential tail $p_k \sim C \exp(-\gamma k)$. Simulations of Chris Varghese suggest that the quadratic contact process on an Erdős–Rényi random graph has a discontinuous transition, but on the power-law random graph in which $p_k = Ck^{-2.5}$ for $k \geq 3$, the critical value is 0 and the transition is continuous.

Our second motivation for exploring particle systems on a random r -regular graph is that it is the natural finite version of a r -tree (in which each vertex has degree r). We think of a random regular graph as a “tree torus”, since the graph looks the same (in distribution) when viewed from any vertex. While the inspiration came from aesthetics, there is a practical consequence: the results for the threshold- θ contact process on the random r -regular graph give as corollaries the corresponding results on r -tree.

1.1. Defining the process on the random graph

In this paper, we study the behavior of the discrete time *threshold- θ contact process* on the random r -regular graph on n vertices. We construct our random graph G_n on the vertex set $V_n := \{1, 2, \dots, n\}$ by assigning r “half-edges” to each of the vertices, and then pairing the half-edges at random. If r is odd, then n must be even so that the number of half-edges, rn , is even to have a valid degree sequence. Let \mathbb{P} denote the distribution of G_n , which is the first of several

probability measures we will define. We condition on the event E_n that the graph is simple, i.e., it does not contain a self-loop at any vertex, or more than one edge between two vertices. It can be shown (see e.g. Corollary 9.7 on p. 239 of [20]) that $\mathbb{P}(E_n)$ converges to a positive limit as $n \rightarrow \infty$, and hence

$$\text{if } \tilde{\mathbb{P}} := \mathbb{P}(\cdot | E_n), \quad \text{then } \tilde{\mathbb{P}}(\cdot) \leq c\mathbb{P}(\cdot) \text{ for some constant } c = c(r) > 0. \quad (2)$$

So the conditioning on the event E_n will not have much effect on the distribution of G_n . It is easy to see that the distribution of G_n under $\tilde{\mathbb{P}}$ is uniform over the collection of all undirected r -regular graphs on the vertex set V_n . We choose G_n according to the distribution $\tilde{\mathbb{P}}$ on simple graphs, and once chosen the graph remains fixed through time.

Having defined the graph, the next step is to define the dynamics on the graph. We write $x \sim y$ to mean that x is a neighbor of y , and let

$$\mathcal{N}_y := \{x \in V_n : x \sim y\} \quad (3)$$

be the set of neighbors of y . The distribution $P_{p,\theta}^{G_n}$ of the (discrete time) threshold- θ contact process $\xi_t \subseteq V_n$ with parameters p and θ conditioned on G_n can be described as follows:

$$\begin{aligned} P_{p,\theta}^{G_n}(x \in \xi_{t+1} \mid |\mathcal{N}_x \cap \xi_t| \geq \theta) &= p \quad \text{and} \\ P_{p,\theta}^{G_n}(x \in \xi_{t+1} \mid |\mathcal{N}_x \cap \xi_t| < \theta) &= 0, \end{aligned}$$

where the decisions for different vertices at time $t + 1$ are taken independently. Let $\xi_t^A \subseteq V_n$ denote the threshold- θ contact process starting from $\xi_0^A = A$, and let ξ_t^1 denote the special case when $A = V_n$.

1.2. Main results

The first step is to prove that threshold- θ contact process dies out for small values of p and survives for p close to 1. It is easy to see that on any graph in which all vertices have degree r the threshold- θ contact process dies out rapidly if $p < 1/r$, because an occupied site has at most r neighbors that it could cause to be occupied at the next time step suggesting $E_{p,\theta}^{G_n} \xi_t^1 \leq n(rp)^t$.

1.2.1. Survival from initial density close to 1

Our next result shows that if $\theta \geq 2$, $r \geq \theta + 2$ and p is sufficiently close to 1, then with high probability the fraction of occupied vertices in the threshold- θ contact process on G_n starting with all 1's stays above $1 - \epsilon_1$ for an exponentially long time.

Theorem 1. Suppose $\theta \geq 2$ and $r \geq \theta + 2$. There are constants $\epsilon_1, \gamma_1 > 0$, and a good set of graphs \mathcal{G}_n with $\mathbb{P}(G_n \in \mathcal{G}_n) \rightarrow 1$ so that if $G_n \in \mathcal{G}_n$ and $p \geq p_1 = 1 - \epsilon_1/(3r - 3\theta)$, then

$$P_{p,\theta}^{G_n} \left(\inf_{t \leq \exp(\gamma_1 n)} \frac{|\xi_t^1|}{n} < 1 - \epsilon_1 \right) \leq \exp(-\gamma_1 n).$$

Here and in what follows, all constants will depend on the degree r and threshold θ . If they depend on other quantities, that will be indicated.

The reason for the restriction to $r \geq \theta + 2$ comes from Proposition 2 (with $j = r - \theta + 1$) below. When $r \leq \theta + 1$, it is impossible to pick $\eta > 0$ so that $(1 + \eta)/(r - \theta) < 1$. There may be more than algebra standing in the way of constructing a proof. We conjecture that the result

is false when $r \leq \theta + 1$. To explain our intuition in the special case $\theta = 2$ and $r = 3$, consider a rooted binary tree in which each vertex has two descendants and hence, except for the root, has degree three. If we start with a density u of 1's on level k and no 1's on levels $m < k$, then at the next step the density will be $g(u) = pu^2 < u$ on level $k - 1$. When each vertex has three descendants instead of two, then

$$g(u) = p(3u^2(1 - u) + u^3),$$

which has a nontrivial fixed point for $p \geq 8/9$ (divide by u and solve the quadratic equation).

As the next result shows, there is a close relationship between the threshold- θ contact process ξ_t on a random r -regular graph and the corresponding process ζ_t on the homogeneous r -tree. Following the standard recipe for attractive interacting particle systems, if we start with all sites on the tree occupied, then the sequence $\{\zeta_t^1\}$ of sets of occupied vertices decreases in distribution to a limit ζ_∞^1 , which is called the *upper invariant measure*, since it is the stationary distribution with the most 1's. Here and later we denote by $\mathbf{0}$ any fixed vertex of the homogeneous tree. Writing $P_{p,\theta}$ for the distribution of ζ_t with parameters p and θ , the critical value is defined by

$$p_c(\theta) := \sup\{p : P_{p,\theta}(\zeta_\infty^1(\mathbf{0}) = 1) = 0\}.$$

Corollary 1. *Suppose $\theta \geq 2$, $r \geq \theta + 2$ and that p_1 and ϵ_1 are the constants in Theorem 1. If $p \geq p_1$, then there is a translation invariant stationary distribution for the threshold- θ contact process on the homogeneous r -tree in which each vertex is occupied with probability $\geq 1 - \epsilon_1$.*

Fontes and Schonmann [12] have considered the continuous time threshold- θ contact process on a tree in which each vertex has degree $b + 1$, and they have shown that if b is large enough, then $\lambda_c < \infty$. Our result improves their result by removing the restriction that b is large.

1.2.2. Dying out from small initial density

If we set the death rate = 0 in the threshold- θ contact process, then we can without loss of generality set the birth rate equal to 1 and the process reduces to bootstrap percolation (with asynchronous updating). Balogh and Pittel [1] have studied bootstrap percolation on random regular graphs. They have identified an interval $[p_-(n), p_+(n)]$ so that the probability that all sites end up active goes sharply from 0 to 1. The limits $p_\pm(n) \rightarrow p_*$ and $p_+ - p_-$ is of order $1/\sqrt{n}$. If bootstrap percolation cannot fill up the graph, then it seems that our process with deaths will be doomed to extinction. The next result proves this, and more importantly extends the result to arbitrary initial conditions with a small density of occupied sites.

Here, since processes with larger θ have fewer survivals, it is enough to prove the result when $\theta = 2$.

Theorem 2. *Suppose $\theta \geq 2$ and $p_2 < 1$. There are constants $0 < \epsilon_2(p_2)$, $C_2(p_2) < \infty$, and a good set of graphs \mathcal{G}_n with $\tilde{\mathbb{P}}(G_n \in \mathcal{G}_n) \rightarrow 1$ so that if $G_n \in \mathcal{G}_n$, then for any $p \leq p_2$, and any subset $A \subset V_n$ with $|A| \leq \epsilon_2 n$,*

$$P_{p,\theta}^{G_n}(\xi_{\lceil C_2 \log n \rceil}^A \neq \emptyset) \leq 2/n^{1/6} \quad \text{for large enough } n.$$

The density of 1's $\rho(p, \theta) := P_{p,\theta}(\zeta_\infty^1(\mathbf{0}) = 1)$ in the stationary distribution on the homogeneous r -tree is a nondecreasing function of p . The next result shows that the threshold- θ contact process on the r -tree has a discontinuous phase transition.

Corollary 2. Suppose $\theta \geq 2$, let p_1 be the constant from Theorem 1, and let $\epsilon_2(\cdot)$ be as in Theorem 2. $\rho(p, \theta)$ never takes values in $(0, \epsilon_2(p_1))$.

This result, like Theorem 2 does not require the assumption $r \geq \theta + 2$. On the other hand, if $\rho(p, \theta) \equiv 0$ for $r \leq \theta + 1$, the result is not very interesting in that case. Again [12] has proved that the threshold- θ contact process has a discontinuous transition when the degree $b + 1$ is large enough.

Fontes and Schonmann [13] have studied θ -bootstrap percolation on trees in which each vertex has degree $b + 1$ and $2 \leq \theta \leq b$. They have shown that there is a critical value p_f so that if $p < p_f$, then for almost every initial configuration of product measure with density p , the final bootstrapped configuration does not have any infinite component. This suggests that we might have $\epsilon_2(p)$ bounded away from 0 as $p \rightarrow 1$.

1.2.3. Stability of 0

The previous pair of results are the most difficult in the paper. From their proofs one easily gets results for the process with spontaneous births with probability β , i.e., after the threshold- θ dynamics has been applied to the configuration at time t , we independently make vacant sites occupied with probability β . For this new process, we denote the set of occupied vertices at time t starting with all 0's by $\hat{\xi}_t^0$ and its distribution conditioned on the graph G_n by $P_{p, \theta, \beta}^{G_n}$ to have the following:

Theorem 3. Suppose $\theta \geq 2$. There is a good set of graphs \mathcal{G}_n with $\tilde{\mathbb{P}}(G_n \in \mathcal{G}_n) \rightarrow 1$ so that if $G_n \in \mathcal{G}_n$ and $p < 1$, then there are constants $C_3(p), \beta_3(p), \gamma_3(p, \beta) > 0$ so that for $\beta < \beta_3$,

$$P_{p, \theta, \beta}^{G_n} \left(\sup_{t \leq \exp(\gamma_3 n)} \frac{|\hat{\xi}_t^0|}{n} > C_3 \beta \right) \leq 2 \exp(-\gamma_3 n).$$

Let $\hat{\xi}_\infty^0$ be the limiting distribution for the process on the homogeneous tree, which exists because of monotonicity.

Corollary 3. If $\theta \geq 2$ and $p < 1$, then $\lim_{\beta \rightarrow 0} P_{p, \theta, \beta}(\hat{\xi}_\infty^0(\mathbf{0}) = 1) = 0$.

1.3. Isoperimetric inequalities

We now describe the “isoperimetric inequalities” that are the keys to the proofs of our results. Let $\partial U := \{y \in U^c : y \sim x \text{ for some } x \in U\}$ be the boundary of U , and given two sets U and W , let $e(U, W)$ be the number of edges having one end in U and the other end in W . Given an $x \in V_n$ let $n_U(x)$ be the number of neighbors of x that are in U , and let

$$U^{*j} = \{x \in V_n : n_U(x) \geq j\}.$$

The estimation of the sizes of $e(U, U^c)$ is an enormous subject with associated key words being Cheeger’s inequality and expander graphs. Bollobás [3] proved the following result for random regular graphs:

Theorem 4. Let $r \geq 3$ and $0 < \eta < 1$ be such that

$$2^{4/r} < (1 - \eta)^{1-\eta} (1 + \eta)^{1+\eta}.$$

Then asymptotically almost surely a random r -regular graph on n vertices has

$$\min_{|U| \leq n/2} \frac{e(U, U^c)}{|U|} \geq (1 - \eta)r/2.$$

To see that the constant is reasonable, choose $n/2$ vertices at random to make U . In this case we expect that $|e(U, U^c)| = nr/2$.

While this result is nice, it is not really useful for us, because we are interested in estimating the size of the boundaries U^{*j} for $j \geq 2$, and in having better constants by only considering small sets.

Proposition 1. Let $E^{*1}(m, \leq k)$ be the event that there is a subset $U \subset V_n$ with size $|U| = m$ so that $|U^{*1}| \leq k$. There are constants C_0 and Δ_0 so that for any $\eta > 0$, there is an $\epsilon_0(\eta)$ which also depends on r so that for $m \leq \epsilon_0(\eta)n$,

$$\mathbb{P} \left[E^{*1}(m, \leq (r - 1 - \eta)m) \right] \leq C_0 \exp \left(-\frac{\eta^2}{4r} m \log(n/m) + \Delta_0 m \right).$$

This result yields the next proposition which we need to prove [Theorems 1](#) and [2](#). For [Theorem 1](#), note that if $W = V_n \setminus \xi_t$ is the set of vacant vertices at time t , then at time $t + 1$ the vertices in $W^{*(r-\theta+1)}$ will certainly be vacant and the vertices in its complement will be vacant with probability $1 - p$. So having an upper bound for $|W^{*(r-\theta+1)}|$ will be helpful. On the other hand for [Theorem 2](#), if U is the set of occupied vertices at time t , then at time $t + 1$ the vertices in $U^{*\theta}$ will be occupied with probability p and the vertices in its complement will certainly be vacant. So having an upper bound for $|U^{*\theta}|$ will be helpful.

Keeping these in mind, it is easy to see from the definitions that if $j > 1$ and $|Z| = m$, then

$$rm \geq \sum_{y \in Z^{*1}} e(\{y\}, Z) \geq |Z^{*1} \setminus Z^{*j}| + j|Z^{*j}| = |Z^{*1}| + (j - 1)|Z^{*j}|.$$

So for any set Z of size m , if $|Z^{*j}| \geq k$, then $|Z^{*1}| \leq rm - (j - 1)k$. Taking $k = m(1 + \eta)/(j - 1)$ so that $rm - (j - 1)k = (r - 1 - \eta)m$ and using [Proposition 1](#) we get the following.

Proposition 2. Let $E^{*j}(m, \geq k)$ be the event that there is a subset $Z \subset V_n$ with size $|Z| = m$ so that $|Z^{*j}| \geq k$. For the constants C_0 , Δ_0 , and $\epsilon_0(\eta)$ given in [Proposition 1](#), if $j > 1$ and $m \leq \epsilon_0(\eta)n$, then

$$\mathbb{P} \left[E^{*j} \left(m, \geq \left(\frac{1 + \eta}{j - 1} \right) m \right) \right] \leq C_0 \exp \left(-\frac{\eta^2}{4r} m \log(n/m) + \Delta_0 m \right).$$

2. Upper bound on the critical value p_c

Proof of Theorem 1. Recall that $r \geq \theta + 2$. Let $\eta = 1/3$ and for ϵ_0, Δ_0 as in [Proposition 1](#) let $\epsilon_1 := \min\{\epsilon_0(\eta), \exp(-8\Delta_0 r/\eta^2)\}$ so that $\log(n/[\epsilon_1 n]) \geq 8\Delta_0 r/\eta^2$ and hence $(\eta^2/4r) \log(n/[\epsilon_1 n]) \geq 2\Delta_0$. With these choices, we apply [Proposition 2](#) with $j = r - \theta + 1$ to have

$$\mathbb{P} \left[E^{*(r-\theta+1)} \left([\epsilon_1 n], \geq \frac{4[\epsilon_1 n]}{3(r - \theta)} \right) \right] \leq C_0 \exp(-\Delta_0 [\epsilon_1 n]).$$

Let $\mathcal{G}_n := E^{*(r-\theta+1)}([\epsilon_1 n], \leq (1+\eta)[\epsilon_1 n]/(r-\theta))$. Since increasing the size of a set U increases $|U^{*\theta}|$, it follows that if $G_n \in \mathcal{G}_n$ and $|U| \geq (1-\epsilon_1)n$, then

$$|U^{*\theta}| \geq \left(1 - \frac{4\epsilon_1}{3(r-\theta)}\right)n.$$

So if $|\xi_t| \geq (1-\epsilon_1)n$ and $p \geq 1 - \epsilon_1/(3r-3\theta)$, then the distribution of $|\xi_{t+1}|$ dominates a

$$\text{Binomial}\left(\left(1 - \frac{4\epsilon_1}{3(r-\theta)}\right)n, p\right) \text{ distribution,}$$

$$\text{which has mean} \geq \left(1 - \frac{5\epsilon_1}{3(r-\theta)}\right)n$$

(the ϵ_1^2 term is positive). When $r \geq \theta + 2$, this is $> (1-\epsilon_1)n$, so standard large deviations for the Binomial distribution imply that there is a constant $\gamma_1(r, \theta) > 0$ so that

$$P_{p,\theta}^{G_n}\left(|\xi_{t+1}| < (1-\epsilon_1)n \mid |\xi_t| \geq (1-\epsilon_1)n\right) \leq \exp(-2\gamma_1 n).$$

If we set $T = \lfloor \exp(\gamma_1 n) \rfloor$, then the probability that $|\xi_{t+1}| \geq (1-\epsilon_1)n$ fails for some $t \leq T$ is $\leq \exp(-\gamma_1 n)$, which completes the proof of [Theorem 1](#). \square

To prepare for the proof of [Corollary 1](#) we need the following result which shows that the random regular graph looks locally like a tree. See e.g., Lemma 2.1 in [\[26\]](#).

Lemma 2.1. Suppose $r \geq 3$ and let $R = \lfloor (1/3) \log_{r-1} n \rfloor$. For any $x \in V_n$, the probability that the collection of vertices in G_n within distance R of x differs from its analogue for $\mathbf{0}$ in the homogeneous r -tree is $\leq 10n^{-1/3}$ for large n .

Proof. Under the law \mathbb{P} , starting with $x \in V_n$ its neighbors in G_n are chosen by selecting r half edges at random from the rn possible options. This procedure continues to select the neighbors of the neighbors, etc. To generate all of the connections out to distance R we will make

$$r[1 + (r-1) + \cdots + (r-1)^{R-1}] \leq rn^{1/3}/(r-2) \quad \text{choices.}$$

The probability that at some point we select a vertex that has already been touched is

$$\leq \frac{rn^{1/3}}{r-2} \cdot \frac{rn^{1/3}/(r-2)}{n - rn^{1/3}/(r-2)} \leq 10n^{-1/3}$$

for large n . \square

Proof of Corollary 1. Let $r \geq \theta + 2$, $p \geq p_1$, and $t(n) = \lfloor \log \log n \rfloor$. To prove that the upper invariant measure is nontrivial we will show that $\lim_{n \rightarrow \infty} P_{p_1, \theta}(\xi_{t(n)}^1(\mathbf{0}) = 1) \geq 1 - \epsilon_1$. To do this note that [Lemma 2.1](#) together with a standard second moment argument applied to H_n = the number of vertices of G_n whose neighborhoods are tree-like up to distance $\lfloor \log \log n \rfloor$ implies that $\tilde{\mathbb{P}}(H_n \leq n - n^{7/8}) \rightarrow 0$. So we can choose $G_n \in \mathcal{G}_n$ having the property that $H_n \geq n - n^{7/8}$. For such a G_n , [Theorem 1](#) implies

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{x=1}^n P_{p_1, \theta}^{G_n}(\xi_{t(n)}^1(x) = 1) \geq 1 - \epsilon_1.$$

Now the state of x at time $t(n)$ can be determined by looking at the values of the process on the space-time cone $\{(y, s) : d(x, y) \leq t(n) - s\}$. Since the space-time cones corresponding to $n - o(n)$ many vertices of G_n are same as that corresponding to $\mathbf{0}$ of the homogeneous r -tree, the desired result follows. \square

3. Extinction from small density, stability of 0

Proof of Theorem 2. We prove the result for $\theta = 2$ only, as the result for $\theta > 2$ follows trivially from the result for $\theta = 2$.

Pick $\eta = \eta(p_2) > 0$ so that $(1 + \eta)p_2 = (1 - 3\eta)$ and then pick $\epsilon_2 := \min\{\epsilon_0(\eta), \exp(-8\Delta_0 r/\eta^2)\}$, where ϵ_0, Δ_0 are as in Proposition 1. So for $m \leq \epsilon_2 n$ we have $(\eta^2/8r) \log(n/m) \geq \Delta_0$. Then using Proposition 2 with $j = 2$,

$$\mathbb{P}[E^{*2}(m, \geq (1 + \eta)m)] \leq C_0 \exp\left(-\frac{\eta^2}{8r} m \log(n/m)\right) \quad \text{for } m \leq \epsilon_2 n.$$

Let \mathcal{G}_n be the collection of graphs so that $E^{*2}(m, \leq (1 + \eta)m)$ holds for all $m \leq \epsilon_2 n$. To see that this event has high probability when n is large, note that

$$\begin{aligned} \mathbb{P}(\mathcal{G}_n^c) &\leq \sum_{m=[n^a]+1}^{[\epsilon_2 n]} C_0 \exp\left(-\frac{\eta^2}{8r} n^a \log(1/\epsilon_2)\right) + \sum_{m=1}^{[n^a]} C_0 \exp\left(-\frac{\eta^2}{8r} \log(n^{1-a})\right) \\ &\leq C_0 \epsilon_2 n \exp\left(-\frac{\eta^2}{8r} n^a \log(1/\epsilon_2)\right) + C_0 n^a n^{-\eta^2(1-a)/8r} \rightarrow 0 \end{aligned}$$

if a is chosen small enough.

As in the proof of Theorem 1, we will use large deviations for the Binomial distribution to control the behavior of the process. However, this time the value of p changes with m , and we will have to stop when the set of occupied vertices gets too small. According to Lemma 2.8.5 in [9].

Lemma 3.1. *If X has Binomial(k, q) distribution, then*

$$P(X \geq k(q + z)) \leq \exp(-kz^2/2(q + z)).$$

Using this result with $k = (1 + \eta)m$ and $q = p_2$ which have $kq = (1 - 3\eta)m$, then taking $z = \eta < 2\eta/(1 + \eta)$ so that $k(q + z) \leq (1 - \eta)m$, it follows that for $m \leq \epsilon_2 n$ and $G_n \in \mathcal{G}_n$,

$$P_{p,2}^{G_n}(|\xi_{t+1}| > (1 - \eta)m \mid |\xi_t| = m) \leq \exp(-\eta^2 m/2(p_2 + \eta)). \quad (4)$$

Using this result $\ell = \lceil (1/2) \log n / (-\log(1 - \eta)) \rceil$ times, we see that if $|\xi_0| \leq \epsilon_2 n$ and $v := \inf\{t : |\xi_t| \leq n^{1/2}\}$, then with high probability $v \leq \ell$.

To finish the process off now we note that when $m \leq \epsilon_2 n$,

$$E_{p,2}^{G_n}(|\xi_{t+1}| \mid |\xi_t| = m) \leq (1 - 3\eta)m. \quad (5)$$

Also note that if $\kappa = \lceil (2/3) \log n / (-\log(1 - 3\eta)) \rceil$ and $G_n \in \mathcal{G}_n$, then

$$|\xi_{v+t}| \leq (1 + \eta)^t n^{1/2} \leq n^{5/6} \quad \text{for } 1 \leq t \leq \kappa, \text{ as } (1 + \eta)^2(1 - 3\eta) < 1 \text{ for any } \eta > 0.$$

So using the inequality in (5) κ times we have $P_{p,2}^{G_n}(|\xi_{v+\kappa}| \geq 1) \leq 1/n^{1/6}$. So combining with (4) we conclude that if $|\xi_0| \leq \epsilon_2 n$ and $G_n \in \mathcal{G}_n$, then

$$P_{p,2}^{G_n}(|\xi_{\kappa+\ell}| \geq 1) \leq 2/n^{1/6} \quad \text{for large enough } n,$$

which proves the desired result with $C_2 = 2/(-\log(1 - \eta))$. \square

Proof of Corollary 2. Suppose $\theta \geq 2$ and that the upper invariant measure for the process on the homogeneous r -tree has $\rho(p, \theta) \in (0, (1 - 3\delta)\epsilon_2(p_1))$ for some $\delta > 0$. It is easy to see that $\epsilon_2(p_1) < 1 - \epsilon_1$, and so it follows from Corollary 1 that $p < p_1$. Pick a time τ so that the threshold- θ contact processes on the homogeneous r -tree has $P_{p,\theta}(\xi_\tau^1(\mathbf{0}) = 1) < (1 - 2\delta)\epsilon_2(p_1)$. The argument involving Lemma 2.1 in the proof of Corollary 1 can be repeated to see that we can choose $G_n \in \mathcal{G}_n$ so that the neighborhoods of $n - o(n)$ many vertices of G_n within distance $\tau + \lceil (\log \log n) / (-\log(1 - \eta(p_1))) \rceil$ look exactly like the analogous neighborhood of $\mathbf{0}$ in the homogeneous r -tree. If n is large, then for the above choices of τ and G_n ,

$$\frac{1}{n} \sum_{x=1}^n P_{p,\theta}^{G_n}(\xi_\tau^1(x) = 1) \leq (1 - \delta)\epsilon_2(p_1).$$

Since the states of the vertices of G_n separated by more than 2τ are independent in ξ_τ^1 , it follows that with $P_{p,\theta}^{G_n}$ -probability tending to 1 as $n \rightarrow \infty$,

$$\sum_{x=1}^n \xi_\tau^1(x) \leq \epsilon_2(p_1)n. \quad (6)$$

Formula (4) implies that after $\ell = \lceil (\log \log n) / (-\log(1 - \eta(p_1))) \rceil$ time units

$$P_{p,\theta}^{G_n} \left(\sum_{x=1}^n \xi_{\tau+\ell}^1(x) \leq n / \log n \right) \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

So by our choice of G_n we have $P_{p,\theta}(\xi_{\tau+\ell}^1(\mathbf{0}) = 1) \leq \rho(p)/2$. Since by monotonicity $P_{p,\theta}(\xi_t^1(\mathbf{0}) = 1)$ is a decreasing function of t , we get a contradiction that proves the desired result. \square

Proof of Theorem 3. Let $\eta(\cdot)$, $\epsilon_2(\cdot)$ and \mathcal{G}_n be as in the proof of Theorem 2, and let U_t be the set of vertices which become occupied at time t because of the dynamics of the threshold- θ contact process. From (4) and a standard large deviations result for the Binomial distribution, it follows that there are constants $\delta_1(p, \beta)$, $\delta_2(p, \beta) > 0$ such that for $m \leq \epsilon_2(p)n$ and $G_n \in \mathcal{G}_n$,

$$\begin{aligned} P_{p,\theta,\beta}^{G_n} \left(|U_{t+1}| > (1 - \eta)m \mid |\hat{\xi}_t^0| = m \right) &\leq \exp(-\delta_1 m), \quad \text{and} \\ P_{p,\theta,\beta}^{G_n} \left(|\hat{\xi}_{t+1}^0| > (1 - \eta)m + 2\beta(n - m) \mid |U_{t+1}| \leq (1 - \eta)m, |\hat{\xi}_t^0| = m \right) \\ &\leq \exp(-\delta_2 n). \end{aligned} \quad (7)$$

Let $\bar{m} = (1 - \eta - 2\beta)m + 2\beta n$, and $\alpha = 2\beta/(\eta + 2\beta)$. If $m = \alpha n$, then $\bar{m} = m$, but doing a little algebra

$$\bar{m} - \alpha n = (1 - \eta - 2\beta)(m - \alpha n),$$

and hence $m \geq 2\alpha n$ implies

$$\bar{m} \leq \left(1 - \frac{\eta + 2\beta}{2} \right) m.$$

We pick $\beta_3 > 0$ small enough so that $\beta < \beta_3$ implies $2\alpha < \epsilon_2(p)/2$. Then for $m = \lceil 2\alpha n \rceil$ we can use the inequalities in (7) to have

$$P_{p,\theta,\beta}^{G_n} \left(|\hat{\xi}_{t+1}^0| > 2\alpha n \mid |\hat{\xi}_t^0| = m \right) \leq 2 \exp(-2\gamma_3 n), \quad \text{where } \gamma_3 := (1/2) \min\{2\alpha\delta_1, \delta_2\}.$$

By monotonicity the above inequality is also true for $m \leq 2\alpha n$. So $|\hat{\xi}_t^0| \leq 2\alpha n$ fails to occur for some $0 \leq t \leq \exp(\gamma_3 n)$ with probability $\leq 2\exp(-\gamma_3 n)$. Hence the desired result follows with $C_3 := 4/\eta$. \square

Proof of Corollary 3. Let β_3, α and \mathcal{G}_n be as in the proof of Theorem 3. It suffices to show that $P_{p,\theta,\beta}(\hat{\xi}_\infty^0(\mathbf{0}) = 1) \leq 5\alpha$ for $\beta < \beta_3$. We prove this by contradiction.

Suppose, if possible, that $P_{p,\theta,\beta}(\hat{\xi}_\infty^0(\mathbf{0}) = 1) > 5\alpha$ for some $\beta \leq \beta_3$. Then there is a time τ at which $P_{p,\theta,\beta}(\hat{\xi}_\tau^0(\mathbf{0}) = 1) \geq 4\alpha$. The argument involving Lemma 2.1 in the proof of Corollary 1 can be repeated to see that we can choose $G_n \in \mathcal{G}_n$ so that the neighborhoods of $n - o(n)$ many vertices of G_n within distance τ look exactly like the analogous neighborhood of $\mathbf{0}$ in the homogeneous r -tree, which implies $\sum_{x=1}^n P_{p,\theta,\beta}^{G_n}(\hat{\xi}_\tau^0(x) = 1) \geq 7\alpha n/2$ for large enough n . Thereafter, imitating the argument which leads to (6) $P_{p,\theta,\beta}^{G_n}(\sum_{x=1}^n \hat{\xi}_\tau^0(x) \geq 3\alpha n) \rightarrow 1$. But then we have a contradiction with the result in Theorem 3. \square

4. Estimates for $e(U, U^c)$ and $|\partial U|$

We begin with a simple estimate for the number of subsets of V_n of size m .

Lemma 4.1. *The number of subsets of V_n of size m is at most $\exp(m \log(n/m) + m)$.*

Proof. The number of subsets of V_n of size m is $\binom{n}{m}$. Using $n(n-1)\cdots(n-m+1) \leq n^m$ and $e^m > m^m/m!$,

$$\binom{n}{m} \leq \frac{n^m}{m!} \leq \left(\frac{ne}{m}\right)^m = \exp(m \log(n/m) + m). \quad \square$$

In order to study the distribution of $|\partial U|$, the first step is to estimate $e(U, U^c)$. Because of the symmetries of our random graph G_n , the distribution of $e(U, U^c)$ under \mathbb{P} depends on U only through $|U|$.

Lemma 4.2. *There are numerical constants $C_{4.2}$ and $\Delta_1 = r(2 + 1/e) + 3/2$ so that if U is a subset of V_n with $|U| = m$ and $\alpha \in [0, 1]$, then*

$$\mathbb{P}(e(U, U^c) \leq \alpha r|U|) \leq C_{4.2} \exp\left(-\frac{r}{2}(1-\alpha)m \log(n/m) + \Delta_1 m\right).$$

Proof. Let $f(u)$ be the number of ways of pairing u objects. Then Stirling's formula $n! \sim (n/e)^n \sqrt{2\pi n}$ implies

$$f(u) = \frac{u!}{(u/2)!2^{u/2}} \sim \sqrt{2}(u/e)^{u/2},$$

and it follows from the limit result that $C_1(u/e)^{u/2} \leq f(u) \leq C_2(u/e)^{u/2}$ for all integers u .

If $q(m, s) = \mathbb{P}(e(U, U^c) = s)$, then we have

$$q(m, s) \leq \binom{rm}{s} \binom{r(n-m)}{s} s! \frac{f(rm-s)f(r(n-m)-s)}{f(rn)}.$$

To see this, recall that we construct the graph G_n by pairing the half-edges at random, which can be done in $f(rn)$ many ways as there are rn half-edges. We can choose the left endpoints of the edges from U in $\binom{rm}{s}$ many ways, the right endpoints from U^c in $\binom{r(n-m)}{s}$ many ways, and

pair them in $s!$ many ways. The remaining $(rm - s)$ many half-edges of U can be paired among themselves in $f(rm - s)$ many ways. Similarly the remaining $(r(n - m) - s)$ many half-edges of U^c can be paired among themselves in $f(r(n - m) - s)$ many ways.

To bound $q(m, s)$ we will use an argument from [9] that begins on the bottom of page 161 and we will follow it until the last display before (6.3.6). To make the connection we note that their $p(m, s) = \binom{n}{m} q(m, s)$ and write $D = rn$, $k = rm$ and $s = \eta k$ for $\eta \in [0, 1]$ to get

$$q(m, s) \leq Ck^{1/2} \left(\frac{e^2}{\eta}\right)^{\eta k} \left(\frac{k}{D}\right)^{k(1-\eta)/2} \left(1 - \frac{(1+\eta)k}{D}\right)^{(D-(1+\eta)k)/2}. \quad (8)$$

A little calculus gives

$$\text{if } \phi(\eta) = \eta \log(1/\eta), \quad \text{then } \phi'(\eta) = -(1 + \log \eta) \text{ and } \phi''(\eta) = -\frac{1}{\eta}. \quad (9)$$

So $\phi(\cdot)$ is a concave function and its derivative vanishes at $1/e$. This shows that the function $\phi(\cdot)$ is maximized at $1/e$, and hence

$$0 \leq \eta \log(1/\eta) \leq 1/e \quad \text{for } \eta \in [0, 1]. \quad (10)$$

So $(e^2/\eta)^{\eta k} \leq B^k$ with $B = \exp(2 + 1/e)$. If we ignore the last term of (8), which is ≤ 1 , then we have

$$\begin{aligned} \mathbb{P}(e(U, U^c) \leq \alpha rm) &\leq \sum_{\{\eta: \eta rm \in \mathbb{N}, \eta \leq \alpha\}} C(rm)^{1/2} B^{rm} \left(\frac{m}{n}\right)^{rm(1-\eta)/2} \\ &\leq Cr^{3/2} m^{3/2} B^{rm} \left(\frac{m}{n}\right)^{r(1-\alpha)m/2}, \end{aligned}$$

as there are at most rm terms in the sum and $(m/n)^{1-\eta} \leq (m/n)^{1-\alpha}$ for $\eta \leq \alpha$. The above bound is

$$\leq C \exp\left(-\frac{r}{2}(1-\alpha)m \log(n/m) + rm \log B + 3m/2\right)$$

and we have the desired result. \square

Lemma 4.2 gives an upper bound for the probability that $e(U, U^c)$ is small. Our next goal is to estimate the difference between $e(U, U^c)$ and $|\partial U|$.

Lemma 4.3. *If U is a subset of vertices of G_n such that $|U| = m$, then there is a constant Δ_2 that depends only on r and an $\epsilon_{4.3}(\eta)$ which also depends on r so that for any $0 < \eta \leq u \leq r$, and $m \leq \epsilon_{4.3}(\eta)n$,*

$$\mathbb{P}(|\partial U| \leq (u - \eta)|U| \mid e(U, U^c) = u|U|) \leq \exp(-\eta m \log(n/m) + \Delta_2 m).$$

Proof. To construct $e(U, U^c)$ we choose um times from the set of $r(n - m)$ half edges attached to U^c . We want to show that with high probability at least $(u - \eta)m$ vertices of U^c are touched. To do this it is enough to show that if the half-edges are chosen one by one, then with high probability at most ηm of them are attached to one of the already touched vertices. We will call the selection of half-edge associated with a vertex that has already been touched a bad choice. At any stage in the process there are at most $(r - 1)um$ bad choices among at least $r(n - m) - um$ choices. Thus the number of bad choices is stochastically dominated by

$$X \sim \text{Binomial}\left(N = um, p = \frac{(r - 1)um}{r(n - m) - um}\right).$$

When $u \leq r$ and $m \leq n/3$, we have $r(n-m) - um \geq r(n-2m) \geq rn/3$ and hence

$$p \leq \frac{(r-1)u}{r/3} \cdot \frac{m}{n} \leq \frac{\eta}{u}$$

when $m \leq \epsilon_{4.3}(\eta)n$.

A standard large deviations result for the Binomial distribution (see e.g., Lemma 2.8.4 in [9]) implies $P(X \geq Nc) \leq \exp(-NH(c))$ for $c > p$, where

$$H(c) = c \log\left(\frac{c}{p}\right) + (1-c) \log\left(\frac{1-c}{1-p}\right). \quad (11)$$

When $c = \eta/u$, the first term in the large deviations bound (11)

$$\begin{aligned} \exp(-Nc \log(c/p)) &\leq \exp\left(-um \cdot \frac{\eta}{u} \left[\log(n/m) + \log(\eta) + \log\left(\frac{r/3}{u^2(r-1)}\right) \right]\right) \\ &\leq \exp[-\eta m \log(n/m) + (m/e) + m\eta \log(3r(r-1))] \end{aligned}$$

by (10). For the second term in the large deviations bound (11) we note that $1/(1-p) > 1$ and use (10) to conclude

$$\exp\left(-N(1-c) \log\left(\frac{1-c}{1-p}\right)\right) \leq \exp(-N(1-c) \log(1-c)) \leq \exp(um/e),$$

which proves the desired result for $\Delta_2 = (r+1)/e + r \log(3r(r-1))$. \square

5. Proof of Proposition 1

We begin by recalling some definitions given earlier and make two new ones. Let $\partial U := \{y \in U^c : y \sim x \text{ for some } x \in U\}$ be the boundary of U , and given disjoint sets U and W let $e(U, W)$ be the number of edges between U and W . Given a vertex x , let $n_U(x)$ be the number of neighbors of x that are in U and let $U^{*1} = \{x \in V_n : n_U(x) \geq 1\}$. Let $U_0 = \{x \in U : n_U(x) = 0\}$ be the set of isolated vertices in U , and let $U_1 = U - U_0$.

Proof. Given $\eta > 0$ define the following events:

$$\begin{aligned} A_U &= \{|U_1| \geq (\eta/2r)|U|\}, \\ B_U &= \{|U^{*1}| \leq (r-1-\eta)|U|\}, \\ D_U &= \{e(U, U^c) \leq (r-2-\eta)|U|\}. \end{aligned} \quad (12)$$

There are three steps in the proof.

- I. Estimate the probability of $F_1 = \cup_{\{U: |U|=m\}} (B_U \cap A_U^c)$.
- II. Estimate the probability of $F_2 = \cup_{\{W: (\eta/2r)m \leq |W| \leq m\}} D_W$.
- III. Estimate the probability of $F_3 = \cup_{\{U: |U|=m\}} B_U \cap F_1^c \cap F_2^c$.

Step I: On the event A_U^c , $|U_0| > (1-\eta/2r)|U|$ and so $e(U, U^c) \geq r|U_0| \geq (r-\eta/2)|U|$. Also on the event B_U , $|\partial U| \leq |U^{*1}| \leq (r-1-\eta)|U|$. From these two observations we have

$$\begin{aligned} \mathbb{P}(B_U \cap A_U^c) &\leq \mathbb{P}(|\partial U| \leq (r-1-\eta)|U|, e(U, U^c) \geq (r-\eta/2)|U|) \\ &\leq \mathbb{P}(e(U, U^c) - |\partial U| \geq (1+\eta/2)|U|). \end{aligned} \quad (13)$$

Combining (13) with the bound in Lemma 4.3, we see that if $|U| = m \leq \epsilon_{4.3}(1+\eta/2)n$, then

$$\mathbb{P}(B_U \cap A_U^c) \leq \exp[-(1+\eta/2)m \log(n/m) + \Delta_2 m]. \quad (14)$$

Using (14) and the inequality in Lemma 4.1 if $m \leq \epsilon_5 n$, then

$$\begin{aligned} \mathbb{P}(F_1) &\leq \binom{n}{m} \exp[-(1 + \eta/2)m \log(n/m) + \Delta_2 m] \\ &\leq \exp[-(\eta/2)m \log(n/m) + (1 + \Delta_2)m]. \end{aligned} \quad (15)$$

If m is small enough, then the above estimate is exponentially small, and so with high probability there is no subset U of size m for which $B_U \cap A_U^c$ occurs.

Step II: Our next step is to estimate the probability that there is a set U of size m for which A_U occurs and $e(U_1, U_1^c) \leq (r - 2 - \eta)|U_1|$. If A_U occurs for some subset U of size m , then $|U_1| \in [\eta m/2r, m]$. Using Lemma 4.2 with $\alpha = 1 - (2 + \eta)/r$ and the inequality in Lemma 4.1,

$$\begin{aligned} \mathbb{P}(F_2) &= \mathbb{P}\left(\bigcup_{m' \in [\eta m/2r, m]} \bigcup_{\{W: |W|=m'\}} \{e(W, W^c) \leq (r - 2 - \eta)m'\}\right) \\ &\leq \sum_{m' \in [\eta m/2r, m]} \binom{n}{m'} C_{4.2} \exp\left[-\left(\frac{2 + \eta}{2}\right)m' \log(n/m') + \Delta_1 m'\right] \\ &\leq \sum_{m' \in [\eta m/2r, m]} C_{4.2} \exp\left[-(\eta/2)m' \log(n/m') + (1 + \Delta_1)m'\right]. \end{aligned} \quad (16)$$

The function $\phi(\eta) = \eta \log(1/\eta)$ is increasing on $(0, 1/e)$ (see (9)), so if $m \leq n/e$ and $m' \in [\eta m/2r, m]$,

$$m' \log(n/m') \geq (\eta m/2r) \log(2rn/\eta m) \geq (\eta/2r)m \log(n/m),$$

since $(\eta/2r) \log(2r/\eta) > 0$. Using the facts that there are fewer than m terms in the sum over m' and the inequality $m \leq e^m$ for $m \geq 0$, we have

$$\mathbb{P}(F_2) \leq C_{4.2} \exp\left(-(\eta^2/4r)m \log(n/m) + (2 + \Delta_1)m\right) \quad (17)$$

when $m \leq n/e$. If m is small enough, then the right-hand side of (17) is exponentially small, and so with high probability there is no subset U of size m for which A_U occurs and $e(U_1, U_1^c) \leq (r - 2 - \eta)|U_1|$.

Step III: Noting that U^{*1} is a disjoint union of U_1 and ∂U we see that if B_U occurs, then

$$(r - 1 - \eta)|U| \geq |U^{*1}| = |U_1| + |\partial U|.$$

Using $|U| = |U_0| + |U_1|$ now we have

$$\Delta(U) \equiv |\partial U| - (r - 2 - \eta)|U_1| - (r - 1 - \eta)|U_0| \leq 0. \quad (18)$$

Also if $|U| = m$, then by the definition of F_1 , $B_U \cap F_1^c \subset B_U \cap A_U$, and on the event $A_U \cap F_2^c$, we have $|U_1| \geq (\eta/2r)|U|$ and $e(U_1, U_1^c) > (r - 2 - \eta)|U_1|$. Combining these observations,

$$\begin{aligned} \mathbb{P}(B_U \cap F_1^c \cap F_2^c) &\leq \mathbb{P}(B_U \cap A_U \cap F_2^c) \\ &\leq \mathbb{P}(\Delta(U) \leq 0, e(U_1, U_1^c) > (r - 2 - \eta)|U_1|). \end{aligned} \quad (19)$$

Now by the definitions of U_0 and U_1 , we have $e(U_0, U^c) = r|U_0|$ and hence

$$e(U, U^c) = r|U_0| + e(U_1, U_1^c), \quad (20)$$

and a little algebra shows that

$$\{\Delta(U) \leq 0\} = \{e(U, U^c) - |\partial U| \geq (1 + \eta)|U_0| + e(U_1, U_1^c) - (r - 2 - \eta)|U_1|\}.$$

Also $e(U_1, U_1^c) < r|U_1|$. So

$$\begin{aligned} & \mathbb{P}(\Delta(U) \leq 0, e(U_1, U_1^c) > (r - 2 - \eta)|U_1|) \\ &= \sum_{\gamma \in (0, 2+\eta)} \mathbb{P}\left(e(U_1, U_1^c) = (r - 2 - \eta + \gamma)|U_1|, e(U, U^c) \right. \\ & \quad \left. - |\partial U| \geq (1 + \eta)|U_0| + \gamma|U_1|\right). \end{aligned} \quad (21)$$

Combining (19) and (21), and recalling that $|U_1| \in [\eta m/2r, m]$,

$$\mathbb{P}(B_U \cap F_1^c \cap F_2^c) = \sum_{\gamma \in (0, 2+\eta)} \sum_{k \in [\eta m/2r, m]} \mathbb{P}(G_{\gamma, k}) \mathbb{P}(H_\gamma | G_{\gamma, k}), \quad (22)$$

where $G_{\gamma, k} = \{e(U_1, U_1^c) = (r - 2 - \eta + \gamma)|U_1|, |U_1| = k\}$ and

$$H_\gamma = \{e(U, U^c) - |\partial U| \geq (1 + \eta)|U_0| + \gamma|U_1|\}.$$

In view of (20), if $R = r - 2 - \eta$ and $L = (R + \gamma)k + r(m - k)$, then $e(U, U^c) = L$ on $G_{\gamma, k}$. So

$$\begin{aligned} \mathbb{P}(H_\gamma | G_{\gamma, k}) &= \mathbb{P}(e(U, U^c) - |\partial U| \geq \gamma k \\ & \quad + (1 + \eta)(m - k) \mid e(U, U^c) = L, |U_1| = k). \end{aligned}$$

Since under the conditional distribution $\mathbb{P}(\cdot | e(U, U^c) = L)$ all the size- L subsets of half-edges corresponding to U^c are equally likely to be paired with those corresponding to U , the conditional distribution of $e(U, U^c) - |\partial U|$ given $e(U, U^c)$ and $|U_1|$ does not depend on $|U_1|$. So we can drop the event $\{|U_1| = k\}$ from the last display and use Lemma 4.3 with η replaced by $\eta' = (\gamma k + (1 + \eta)(m - k))/m$ to see that if $m \leq \epsilon_{4.3}(\eta')n$, then

$$\mathbb{P}(H_\gamma | G_{\gamma, k}) \leq \exp(-\{\gamma k + (1 + \eta)(m - k)\} \log(n/m) + \Delta_2 m). \quad (23)$$

In order to estimate $\mathbb{P}(G_{\gamma, k})$, we again use (20) to conclude

$$\begin{aligned} \mathbb{P}(G_{\gamma, k}) &= \mathbb{P}(e(U_1, U_1^c) = (r - 2 - \eta + \gamma)k, |U_1| = k) \\ &= \mathbb{P}(e(U, U^c) = (r - 2 - \eta + \gamma)k + r(m - k), |U_1| = k) \\ &\leq \mathbb{P}(e(U, U^c) = rm - (2 + \eta - \gamma)k). \end{aligned}$$

Using Lemma 4.2 with $\alpha = 1 - (2 + \eta - \gamma)k/rm$,

$$\mathbb{P}(G_{\gamma, k}) \leq C_{4.2} \exp\left(-\frac{2 + \eta - \gamma}{2} k \log(n/m) + \Delta_1 m\right). \quad (24)$$

Combining (22)–(24) if $m \leq \epsilon_0(\eta)n$, where $\epsilon_0(\eta) = \min\{\epsilon_{4.3}(1 + \eta/2), \epsilon_{4.3}(\eta')\}$, then

$$\begin{aligned} \mathbb{P}(B_U \cap F_1^c \cap F_2^c) &\leq \sum_{\gamma \in (0, 2+\eta)} \sum_{k \in [\eta m/2r, m]} C_{4.2} \exp((\Delta_1 + \Delta_2)m) \\ & \quad \times \exp\left[-\left\{\left(1 + \frac{\eta + \gamma}{2}\right)k + (1 + \eta)(m - k)\right\} \log(n/m)\right]. \end{aligned} \quad (25)$$

To simplify the second exponential we drop the $\gamma/2$ from the first term and reduce the η to $\eta/2$ in the second in order to combine them into $(1 + \eta/2)m$. Noting that there are fewer than rm terms in the sum over γ and at most m terms in the sum over k , and using the inequality $m^2 \leq e^m$

for $m \geq 0$, the above is

$$\begin{aligned} &\leq C_{4.2} r m^2 \exp \left[-(1 + \eta/2)m \log(n/m) + (\Delta_1 + \Delta_2)m \right] \\ &\leq C_{4.2} r \exp \left[-(1 + \eta/2)m \log(n/m) + (1 + \Delta_1 + \Delta_2)m \right]. \end{aligned} \quad (26)$$

Recalling that $E^{*1}(m, \leq (r - 1 - \eta)m) = \cup_{\{U: |U|=m\}} B_U$ we have

$$\mathbb{P}(E^{*1}(m, \leq (r - 1 - \eta)m)) \leq \mathbb{P}(F_1) + \mathbb{P}(F_2) + \sum_{\{U: |U|=m\}} \mathbb{P}(B_U \cap F_1^c \cap F_2^c).$$

Combining (15), (17) and (26), and using $\binom{n}{m} \leq \exp(m \log(n/m) + m)$ from Lemma 4.1 we see that the above is

$$\begin{aligned} &\leq \exp[-(\eta/2)m \log(n/m) + (1 + \Delta_2)m] \\ &\quad + C_{4.2} \exp \left[-(\eta^2/4r)m \log(n/m) + (2 + \Delta_1)m \right] \\ &\quad + C_{4.2} r \exp \left[-(\eta/2)m \log(n/m) + (2 + \Delta_1 + \Delta_2)m \right] \\ &\leq C \exp[-(\eta^2/4r)m \log(n/m) + (2 + \Delta_1 + \Delta_2)m], \end{aligned} \quad (27)$$

which is the desired result. \square

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